ON THE CURVATURE OF SYMMETRIC PRODUCTS OF A COMPACT RIEMANN SURFACE

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Abstract. Let X be a compact connected Riemann surface of genus at least two. The main theorem of [BR] says that for any positive integer $n \leq 2(\text{genus}(X)-1)$, the symmetric product $S^n(X)$ does not admit any Kähler metric satisfying the condition that all the holomorphic bisectional curvatures are nonnegative. Our aim here is to give a very simple and direct proof of this result of Bökstedt and Romão.

1. Introduction

The main theorem of [BR] says the following (see [BR, Theorem 1.1]):

Let X be a compact connected Riemann surface of genus at least two. If

$$1 \leq n \leq 2(\text{genus}(X)-1)\,,$$

then the symmetric product $S^n(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative.

Our aim here is to give a simple proof of this theorem.

In [Bi], the following related was proved (see [Bi, Theorem 1.1]):

Let X be a compact connected Riemann surface of genus at least two. If

$$n > 2(\text{genus}(X)-1)\,,$$

then $S^n(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative.

2. Nonnegative holomorphic bisectional curvature

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. For any positive integer n, the n-fold symmetric product of X will be denoted by $S^n(X)$. We recall that $S^n(X)$ is the quotient of X^n by the natural action of the group of permutations P_n of the index set $\{1\,,\cdots,n\}$. Let

$$q\,:\,X^n\,\longrightarrow\,S^n(X)\,:=\,X^n/P_n \tag{2.1}$$

be the quotient map.

Theorem 2.1. *Take any integer $n \in [1\,,2(g-1)]$. The symmetric product $S^n(X)$ does not admit any Kähler metric satisfying the condition that all the holomorphic bisectional curvatures are nonnegative.*

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Proof. Let $K_{X^n}^{-1} = \bigwedge^n TX^n$ and $K_{S^n(X)}^{-1} = \bigwedge^n TS^n(X)$ be the anti-canonical line bundles of X^n and $S^n(X)$ respectively. Fix distinct $n-1$ points $\{x_1, \cdots, x_{n-1}\}$ of X. Let

$$\varphi : X \longrightarrow X^n \tag{2.2}$$

be the embedding defined by $x \longmapsto \{x, x_1, \cdots, x_{n-1}\} \in X^n$. Define

$$\widetilde{\varphi} := q \circ \varphi : X \longrightarrow S^n(X), \tag{2.3}$$

where q is defined in (2.1).

We will show that

$$\text{degree}(\widetilde{\varphi}^* K_{S^n(X)}^{-1}) = n - 2g + 1. \tag{2.4}$$

For $1 \leq i < j \leq n$, let $D_{i,j} \subset X^n$ be the divisor consisting of all $\{y_1, \cdots, y_n\}$ such that $y_i = y_j$. Let

$$D := \sum_{1 \leq i < j \leq n} D_{i,j}$$

be the divisor on X^n. For the map q in (2.1), we have

$$q^* K_{S^n(X)}^{-1} = K_{X^n}^{-1} \otimes \mathcal{O}_{X^n}(D).$$

Therefore,

$$\widetilde{\varphi}^* K_{S^n(X)}^{-1} = \varphi^* q^* K_{S^n(X)}^{-1} = (\varphi^* K_{X^n}^{-1}) \otimes (\varphi^* \mathcal{O}_{X^n}(D)), \tag{2.5}$$

where φ is the map in (2.2).

Now,

$$\varphi^* K_{X^n}^{-1} = TX \quad \text{and} \quad \varphi^* \mathcal{O}_{X^n}(D) = \mathcal{O}_X(\sum_{i=1}^{n-1} x_i),$$

where TX is the holomorphic tangent bundle of X. Therefore, from (2.5) we conclude that

$$\widetilde{\varphi}^* K_{S^n(X)}^{-1} = (TX) \otimes \mathcal{O}_X(\sum_{i=1}^{n-1} x_i).$$

Hence

$$\text{degree}(\widetilde{\varphi}^* K_{S^n(X)}^{-1}) = 2 - 2g + n - 1 = n - 2g + 1,$$

and (2.4) is proved.

Since $n \leq 2g - 2$, from (2.4) we conclude that

$$\text{degree}(\widetilde{\varphi}^* K_{S^n(X)}^{-1}) < 0.$$

This immediately implies that $S^n(X)$ does not admit a Kähler metric for which all the holomorphic bisectional curvatures are nonnegative. $\square$

The Chern class $c_1(K_{S^n(X)}^{-1})$ is computed in [Ma, p. 333] and [ACGH, p. 323]. It should be possible to derive (2.4) using this description of $c_1(K_{S^n(X)}^{-1})$.

References

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